SHARP LIEB-THIRRING INEQUALITIES IN HIGH DIMENSIONS

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ABSTRACT. We show how a matrix version of the Buslaev-Faddeev-Zakharov trace formulae for a one-dimensional Schrödinger operator leads to Lieb-Thirring inequalities with sharp constants $L^{\mathrm{cl}}_{\gamma,d}$ with $\gamma \geq 3/2$ and arbitrary $d \geq 1$.

0. INTRODUCTION

Let us consider a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$-\Delta + V\,, \tag{0.1}$$

where V is a real-valued function. In [22] Lieb and Thirring proved that if $\gamma > \max(0, 1 - d/2)$, then there exist universal constants $L_{\gamma,d}$ satisfying[1]

$$\operatorname{tr}\,(-\Delta + V)_-^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}(x)\,dx\,. \tag{0.2}$$

In the critical case $d \geq 3$ and $\gamma = 0$ the bound (0.2) is known as the Cwikel-Lieb-Rozenblum (CLR) inequality, see [7, 19, 24] and also [6, 18]. For the remaining case $d = 1$, $\gamma = 1/2$ the estimate (0.2) has been verified in [26], see also [13]. On the other hand it is known that (0.2) fails for $\gamma = 0$ if $d = 2$ and for $0 \leq \gamma < 1/2$ if $d = 1$.

If $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$, then the inequalities (0.2) are accompanied by the Weyl type asymptotic formula

$$\begin{aligned}\lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \operatorname{tr}\,(-\Delta + \alpha V)_-^\gamma &= \lim_{\alpha\to+\infty} \frac{1}{\alpha^{\gamma+\frac{d}{2}}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 + \alpha V)_-^\gamma \frac{dx\,d\xi}{(2\pi)^d} \\ &= L^{\mathrm{cl}}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\,dx\,, \end{aligned} \tag{0.3}$$

1991 *Mathematics Subject Classification.* Primary 35P15; Secondary 35L15, 47A75, 35J10.

[1]Here and below we use the notion $2x_- := |x| - x$ for the negative part of variables, functions, Hermitian matrices or self-adjoint operators.

where the so-called classical constant $L^{\mathrm{cl}}_{\gamma,d}$ is defined by

$$L^{\mathrm{cl}}_{\gamma,d} = (2\pi)^{-d} \int_{\mathbb{R}^d} (|\xi| - 1)^{\gamma}_{-}\, d\xi = \frac{\Gamma(\gamma+1)}{2^d \pi^{d/2} \Gamma(\gamma + \frac{d}{2} + 1)}\,, \quad \gamma \geq 0\,. \tag{0.4}$$

It is interesting to compare the value of the sharp constant $L_{\gamma,d}$ in (0.2) and the value of $L^{\mathrm{cl}}_{\gamma,d}$. In particular, the asymptotic formula (0.3) implies that

$$L^{\mathrm{cl}}_{\gamma,d} \leq L_{\gamma,d} \tag{0.5}$$

for all d and γ whenever (0.2) holds. Moreover, in [1] it has been shown, that for a fixed d the ratio $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is a monotone non-increasing function of γ. In conjunction with the Buslaev-Faddeev-Zakharov trace formulae [5, 8] one obtains [22]

$$L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d} \tag{0.6}$$

for

$$d = 1 \qquad \text{and} \qquad \gamma \geq 3/2\,. \tag{0.7}$$

On the other hand one knows that

$$L^{\mathrm{cl}}_{\gamma,d} < L_{\gamma,d}$$

if $d = 1$ and $1/2 \leq \gamma < 3/2$ (see [22]) or $\gamma < 1$ and $d \in \mathbb{N}$ (see [11]).

Up to now (0.7) was the only case where (0.6) was known to be true for general classes of potentials $V \in L^{\gamma + \frac{d}{2}}$. Notice, however, that (0.6) has been proven for various *subclasses* of potentials. If, for example, $\Omega \subset \mathbb{R}^d$ is a domain of finite measure,

$$V(x) = \begin{cases} -\alpha & \text{as} \quad x \in \Omega \\ \infty & \text{as} \quad x \in \mathbb{R}^d \setminus \Omega \end{cases},$$

then the equality (0.6) with $\gamma = 0$ can be identified with the Pólya conjecture on the number of the eigenvalues less than α for the Dirichlet Laplacian in Ω. It holds true for tiling domains [23] and has been justified in [15] for certain domains of product structure by using the method of “lifting” with respect to the dimension d which is also one of the main ideas of this paper. If $\gamma \geq 1$, then (0.6) is true for arbitrary Ω. This is a simple corollary of the Berezin-Lieb inequality (see [2], §5; [20] and also [17]). This approach has been extended in [15] to the Dirichlet boundary value problems for matrices of pseudodifferential operators in $\mathbb{R}^d$ with constant coefficients. The Berezin-Lieb inequality was also used in [16] in order to improve the Lieb constant [19] in the CLR inequality for the subclass of Schrödinger operators whose potentials are equal to the characteristic functions of sets of finite measure.

Another example is given in [4], where the identity (0.6) with $\gamma \geq 1$ and $d \in \mathbb{N}$ has been verified for a class of quadratic potentials.

We note, that with the exception of (0.7), the sharp value of $L_{\gamma,d}$ has been recently found in [13], where it was proved that for $d = 1$ and $\gamma = 1/2$

$$L_{1/2,1} = 2L^{cl}_{1/2,1} = 1/2\,.$$

In particular, in higher dimensions $d \geq 2$ the sharp values of the constants $L_{\gamma,d}$ have been unknown.

The main purpose of this paper is to verify (0.6) for any $\gamma \geq 3/2$, $d \in \mathbb{N}$ and any $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$.

In fact, this result is obtained for infinite-dimensional systems of Schrödinger equations. Let $\mathbf{G}$ be a separable Hilbert space, let $\mathbf{1}_{\mathbf{G}}$ be the identity operator on $\mathbf{G}$ and consider

$$-\Delta \otimes \mathbf{1}_{\mathbf{G}} + V(x)\,, \qquad x \in \mathbb{R}^d\,, \tag{0.8}$$

in $L^2(\mathbb{R}^d, \mathbf{G})$. Here $V(x)$ is a family of self-adjoint non-positive operators in $\mathbf{G}$, such that tr $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$. Then we prove that

$$\mathrm{tr}\ (-\Delta \otimes \mathbf{1}_{\mathbf{G}} + V(x))^{\gamma}_{-} \leq L^{cl}_{\gamma,d} \int_{\mathbb{R}^d} \mathrm{tr}\ V_{-}^{\gamma+\frac{d}{2}}(x)\, dx \tag{0.9}$$

for all $\gamma \geq 3/2$ and $d \geq 1$. The inequality (0.9) can be extended to magnetic Schrödinger operators and we apply it to the Pauli operator.

We shall first deduce (0.9) for $d = 1$, $\gamma = 3/2$ and $\mathbf{G} = \mathbb{C}^n$ from the appropriate trace formula (1.61) for a finite system of one-dimensional Schrödinger operators. In the scalar case these trace identities are known as Buslaev-Faddeev-Zakharov formulae [5, 8]. The matrix case can be handled in a similar way as in the scalar case (see [8]) However, we give rather complete proofs of the corresponding statements in section 1, since we were unable to find the necessary formula (1.61) in the numerous papers devoted to this subject.

Note that we discuss trace formulae only as a technical tool in order to establish bounds on the negative spectrum. We therefore develop the theory of trace identities only as far as it is necessary for our own purpose.

In section 2 we extend the results of section 1 to the Schrödinger operator in $L^2(\mathbb{R}^1, \mathbf{G})$. Applying a "lifting" argument with respect to dimension as used in [9] and [15], we obtain in section 3 the main results of this paper.

Finally we would like to notice that the combination of the results of this paper and the equality $L_{1/2,1} = 1/2$ discovered in [13] has lead to new bounds on the Lieb-Thirring constants in [12] which improve the corresponding bound obtained in [3] and [21].

1. Trace formulae for elliptic systems

1.1. **Jost Functions.** Let 0 and 1 be the zero and the identity operator on $\mathbb{C}^n$. We consider the system of ordinary differential equations

$$-\left(\frac{d^2}{dx^2}\otimes 1\right)y(x)+V(x)y(x)=k^2y(x)\,,\qquad x\in\mathbb{R}\,, \tag{1.1}$$

where V is a compactly supported, smooth (not necessary sign definite) Hermitian matrix-valued function. Define

$$x_{\min}:=\min\operatorname{supp}V\quad\text{and}\quad x_{\max}:=\max\operatorname{supp}V\,.$$

Then for any $k\in\mathbb{C}\backslash\{0\}$ there exist unique $n\times n$ matrix-solutions $F(x,k)$ and $G(x,k)$ of the equations

$$-F''_{xx}(x,k)+VF(x,k)=k^2F(x,k)\,, \tag{1.2}$$

$$-G''_{xx}(x,k)+VG(x,k)=k^2G(x,k)\,, \tag{1.3}$$

satisfying

$$F(x,k)=e^{ikx}1\qquad\text{as}\quad x\geq x_{\max}\,, \tag{1.4}$$

$$G(x,k)=e^{-ikx}1\quad\text{as}\quad x\leq x_{\min}\,. \tag{1.5}$$

If $k\in\mathbb{C}\setminus\{0\}$, then the pairs of matrices $F(x,k)$, $F(x,-k)$ and $G(x,k)$, $G(x,-k)$ form full systems of independent solutions of (1.1). Hence the matrix $F(x,k)$ can be expressed as a linear combination of $G(x,k)$ and $G(x,-k)$

$$F(x,k)=G(x,k)B(k)+G(x,-k)A(k) \tag{1.6}$$

and vice versa:

$$G(x,k)=F(x,k)\beta(k)+F(x,-k)\alpha(k)\,. \tag{1.7}$$

1.2. **Basic properties of the matrices $A(k)$, $B(k)$, $\alpha(k)$ and $\beta(k)$ for real k.** Throughout this subsection we assume that $k\in\mathbb{R}\backslash\{0\}$. Consider the Wronskian type matrix function

$$W_1[F,G](x,k)=G^*(x,k)F'_x(x,k)-(G'_x(x,k))^*F(x,k)\,.$$

Then by (1.2) and (1.3) for $k\in\mathbb{R}$ we find that

$$\frac{d}{dx}W_1[F,G](x,k)=G^*(x,k)F''_x(x,k)-(G''_x(x,k))^*F(x,k)=0\,.$$

Note that for $x\leq x_{\min}$ by (1.6) we have

$$\begin{aligned}W_1[F,G](x,k)&=[G^*(x,k)G'_x(x,k)-(G'_x(x,k))^*G(x,k)]\,B(k)\\&\qquad+[G^*(x,k)G'_x(x,-k)-(G'_x(x,k))^*G(x,-k)]\,A(k)\\&=-2ikB(k)\,,\end{aligned}$$

while for $x \geq x_{\max}$ by (1.7) we find

$$\begin{aligned} W_1[F,G](x,k) &= \beta^*(k)\,[F^*(x,k)F'_x(x,k) - (F'_x(x,k))^*F(x,k)] \\ &\quad + \alpha^*(k)\,[F^*(x,-k)F'_x(x,k) - (F'_x(x,-k))^*F(x,k)] \\ &= 2ik\beta^*(k)\,. \end{aligned}$$

This allows us to conclude that

$$\beta^*(k) = -B(k)\,. \tag{1.8}$$

Similarly, for the matrix-valued function

$$W_2[F,G](x,k) = G^*(x,k)F'_x(x,-k) - (G'_x(x,k))^*F(x,-k)$$

we have $\frac{d}{dx}W_2[F,G](x,k) = \mathbf{0}$ and

$$\begin{aligned} W_2[F,G](x,k) &= -2ikA(-k) \qquad \text{as} \qquad x \leq x_{\min}\,, \\ W_2[F,G](x,k) &= -2ik\alpha^*(k) \qquad \text{as} \qquad x \geq x_{\max}\,. \end{aligned}$$

Thus,

$$A(-k) = \alpha^*(k)\,. \tag{1.9}$$

Inserting (1.6) into (1.7) and making use of (1.8), (1.9) we obtain

$$\begin{aligned} G(x,k) &= G(x,k)\,[B(k)\beta(k) + A(-k)\alpha(k)] \\ &\quad + G(x,-k)\,[A(k)\beta(k) + B(-k)\alpha(k)]\,, \end{aligned} \tag{1.10}$$

and thus

$$A(-k)A^*(-k) - B(k)B^*(k) = \mathbf{1}\,, \tag{1.11}$$

$$B(-k)A^*(-k) - A(k)B^*(k) = \mathbf{0}\,. \tag{1.12}$$

In particular, this implies

$$|\det A(k)|^2 = \det A(k)\det A^*(k) = \det(\mathbf{1} + B(-k)B^*(-k)) \geq 1 \tag{1.13}$$

for all $k \in \mathbb{R}\setminus\{0\}$.

1.3. **Associated Volterra equations and auxiliary estimates.** Next we derive estimates for the fundamental solutions of (1.1) for $\operatorname{Im} k \geq 0$. Note first that the matrices $F(x,k)$ and $G(x,k)$ are solutions of the integral equations

$$F(x,k) = e^{ikx}\mathbf{1} - \int_x^\infty k^{-1}\sin k(x-t)V(t)F(t,k)\,dt\,, \tag{1.14}$$

$$G(x,k) = e^{-ikx}\mathbf{1} + \int_{-\infty}^x k^{-1}\sin k(x-t)V(t)G(t,k)\,dt\,. \tag{1.15}$$

Put

$$H(x,k) = e^{-ikx}F(x,k) - \mathbf{1}\,.$$

Obviously, this matrix-valued function satisfies

$$H(x,k) = 0 \quad \text{for} \quad x \geq x_{\max} \tag{1.16}$$

and

$$H(x,k) = \int_x^\infty K(x,t,k)\,dt + \int_x^\infty K(x,t,k)H(t,k)\,dt, \tag{1.17}$$

where

$$K(x,t,k) = \frac{e^{2ik(t-x)} - 1}{2ik} V(t)\,. \tag{1.18}$$

Note that

$$\|K(x,t,k)\| \leq C_1(V,n)/(1+|k|) \tag{1.19}$$

for all k with Im $k \geq 0$ and all k with $x_{\min} \leq x \leq t$. Here and below $\|\cdot\|$ denotes the norm of a matrix on $\mathbb{C}^n$.

Solving the Volterra equation (1.17) we obtain the convergent series

$$H(x,k) = \sum_{m=1}^{\infty} \int \cdots \int_{x \leq x_1 \leq \cdots \leq x_m} \prod_{l=1}^{m} K(x_{l-1}, x_l, k)\,dx_1 \cdots dx_m\,.$$

From (1.19) we see that $|H(x,k)| \leq C_2(V)$ for all $x_{\min} \leq x \leq x_{\max}$. Inserting this estimate back into (1.17), we conclude that the inequality

$$\|H(x,k)\| \leq C_3(V,n)(1+|k|)^{-1} \tag{1.20}$$

holds for all x with $x_{\min} \leq x \leq x_{\max}$ and all k with Im $k \geq 0$.

Remark 1.1. If we assume that Im $k \geq 0$ and $|k| \geq 1$, then (1.19) and therefore (1.20) holds true for all $x \in \mathbb{R}$.

It is not difficult to observe, that $H(x,k)$ defined by (1.17) is smooth in

$$(x,k) \in \mathbb{R} \times \{k \in \mathbb{C} : \text{Im } k \geq 0\}\,.$$

In particular, if we differentiate (1.17) with respect to $\overline{k}$ we find that

$$\frac{\partial}{\partial \overline{k}} H(x,k) = \int_x^\infty K(x,t,k) \frac{\partial}{\partial \overline{k}} H(t,k)\,dt\,.$$

Since $\partial H(x,k)/\partial\overline{k}$ satisfies a homogeneous Volterra integral equation with the kernel (1.18), we obtain $\partial H(x,k)/\partial\overline{k} \equiv 0$, and thus all the entries of the matrix $H(x,k)$ are analytic in k for Im $k > 0$.

1.4. **Further Estimates on $A(k)$ and $B(k)$.** If we rewrite (1.14) as follows

$$F(x,k) = e^{ikx}\left[\mathbf{1} - \frac{1}{2ik}\int_x^\infty V(t)\,dt - \frac{1}{2ik}\int_x^\infty V(t)H(t,k)\,dt\right] + \frac{e^{-ikx}}{2ik}\left[\int_x^\infty e^{2ikt}V(t)\,dt + \int_x^\infty e^{2ikt}V(t)H(t,k)\,dt\right], \tag{1.21}$$

then the expressions in the brackets in the r.h.s. do not depend on x for $x \le x_{\min}$. Comparing (1.21) with (1.6) we see that

$$A(k) = \mathbf{1} - \frac{1}{2ik}\int_{-\infty}^{+\infty} V(t)\,dt - \frac{1}{2ik}\int_{-\infty}^{+\infty} V(t)H(t,k)\,dt\,, \tag{1.22}$$

$$B(k) = \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2ikt}V(t)\,dt + \frac{1}{2ik}\int_{-\infty}^{+\infty} e^{2ikt}V(t)H(t,k)\,dt\,. \tag{1.23}$$

For sufficiently large $|k| > C$ the smoothness of V and (1.20) imply

$$\left\| A(k) - \mathbf{1} + \frac{1}{2ik}\int_{-\infty}^{+\infty} V(t)dt \right\| \le C_4(V,n)|k|^{-2}, \quad \mathrm{Im}\, k \ge 0\,, \tag{1.24}$$

$$\|B(k)\| \le C_5(V,n)|k|^{-2}\,, \quad k \in \mathbb{R}\,. \tag{1.25}$$

In subsection 1.6 we shall see that (1.25) can be improved so that

$$B(k) = O(|k|^{-m}) \quad \text{for all} \quad m \in \mathbb{N} \quad \text{as} \quad k \to \pm\infty\,. \tag{1.26}$$

1.5. **The matrix $A(k)$ for Im $k \ge 0$.** First note that all entries of the matrix $A(k)$ are analytic in k for Im $k > 0$ and continuous for Im $k \ge 0$, $k \ne 0$. This follows from (1.22) and the analyticity of $H(x,k)$. Fixing a sufficiently small $\epsilon > 0$ and by using (1.22) and (1.20) we obtain

$$\|A(k)\| \le C_6|k|^{-1} \qquad \text{as} \qquad |k| < \epsilon, \quad \mathrm{Im}\, k \ge 0. \tag{1.27}$$

Moreover, all the entries of $A(k)$ and thus the function $\det A(k)$ are analytic for Im $k > 0$ and continuous for Im $k \ge 0$, $k \ne 0$. Near the point $k = 0$ we find

$$|\det A(k)| \le C_7|k|^{-n} \qquad \text{as} \qquad |k| < \epsilon, \quad \mathrm{Im}\, k \ge 0. \tag{1.28}$$

Next let us describe the connection between the function $\det A(k)$ and the spectral properties of the self-adjoint problem (1.1) on $L^2(\mathbb{R},\mathbb{C}^n)$. Our assumptions on the matrix potential V imply, that the operator on the l.h.s. of (1.1) has a discrete negative spectrum, which consists of finitely many negative eigenvalues $\lambda_l = (i\varkappa_l)^2$, $\varkappa_l > 0$ of finite multiplicities m_l. Obviously a solution $y(x)$ of (1.1) with $k = i\varkappa_l$ belongs to $L^2(\mathbb{R},\mathbb{C}^n)$, if and only if

$$y(x) = G(x, i\varkappa_l)e_y^G \quad \text{as} \quad x \le x_{\min}\,,$$
$$y(x) = F(x, i\varkappa_l)e_y^F \quad \text{as} \quad x \ge x_{\max}\,,$$

for some non-trivial vectors $e_G, e_F \in \mathbb{C}^n$. Linear independent solutions $y_1, \dots, y_{m_l}$ define linear independent vectors $e^G_{y_1}, \dots, e^G_{y_{m_l}}$ and $e^F_{y_1}, \dots, e^F_{y_{m_l}}$, respectively. In view of (1.6) we conclude that

$$\dim\ker A(i\varkappa_l) = m_l\,. \tag{1.29}$$

If we select an orthonormal basis in $\mathbb{C}^n$, such that the first m_l elements belong to $\ker A(i\varkappa_l)$, we find that the first m_l rows of $A(k)$ vanish as $k \to i\varkappa_l$. Since $\det A(k)$ does not depend on the choice of the orthonormal basis and all entries of $A(k)$ are analytic, the function $\det A(k)$ has a zero of the order

$$m_l' \geq m_l \tag{1.30}$$

at $k = i\varkappa_l$, $\varkappa_l > 0$. Moreover, if $\lambda = k^2$, $\mathrm{Im}\, k > 0$ is not an eigenvalue of the problem (1.1), then $\det A(k) \neq 0$.

In the remaining part of this subsection we prove that

$$m_l' = m_l\,. \tag{1.31}$$

Let $g(x, y, k)$ be the Green function of the problem (1.1). If $k^2 < 0$, $\mathrm{Im}\, k > 0$, and $\det A(k) \neq 0$ it can be written as

$$g(x, y, k) = \begin{cases} G(x, k)Z^-(y, k) & \text{as} \quad y > x \\ -F(x, k)Z^+(y, k) & \text{as} \quad y < x \end{cases}.$$

Here $Z^+(y, k)$ and $Z^-(y, k)$ are $n \times n$-matrices, which are chosen such that

$$\lim_{x=y-0} g(x, y; k) = \lim_{x=y+0} g(x, y; k)\,,$$
$$\lim_{x=y-0} g_x'(x, y; k) = \lim_{x=y+0} g_x'(x, y; k) + \mathbf{1}\,.$$

These equations turn into

$$W(y, k)\begin{pmatrix} Z^-(y, k) \\ Z^+(y, k) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix}, \quad W(y, k) = \begin{pmatrix} G(y, k) & F(y, k) \\ G_y'(y, k) & F_y'(y, k) \end{pmatrix}. \tag{1.32}$$

Since $\frac{\partial}{\partial y}\det W = \mathbf{0}$, the determinant of W is a constant with respect to y. If y with $y < x_{\min}$, $\mathrm{Im}\, k > 0$, then in view of (1.6) and (1.5) we have

$$W(y, k) = \begin{pmatrix} e^{-iky}\mathbf{1} & e^{-iky}B(k) + e^{iky}A(k) \\ -ike^{-iky}\mathbf{1} & -ike^{-iky}B(k) + ike^{iky}A(k) \end{pmatrix}. \tag{1.33}$$

Hence

$$\det W = (2ik)^n \det A(k)$$

and W is invertible if and only if $\det A(k) \neq 0$. From (1.33) we see then, that for $y < x_{\min}$ the entries X_{ij} of

$$W^{-1}(y,k) = \begin{pmatrix} X_{11}(y,k) & X_{12}(y,k) \\ X_{21}(y,k) & X_{22}(y,k) \end{pmatrix} \tag{1.34}$$

satisfy

$$\begin{aligned} e^{-iky}X_{21} - ike^{-iky}X_{22} &= \mathbf{0}\,, \\ e^{-iky}(X_{21} - ikX_{22})B(k) + e^{iky}(X_{21} + ikX_{22})A(k) &= \mathbf{1}\,. \end{aligned}$$

This gives $X_{21}(y,k) = ikX_{22}(y,k)$, and thus

$$X_{22}(y,k) = (2ik)^{-1}e^{-iky}A^{-1}(k)\,.$$

In view of (1.32) and (1.34) we obtain $Z^+(y,k) = X_{22}(y,k)$ and finally conclude that

$$g(x,y,k) = -(2ik)^{-1}A^{-1}(k)e^{ik(x-y)} \quad \text{as} \quad y < x_{\min} < x_{\max} < x\,. \tag{1.35}$$

If k is in a sufficiently small neighbourhood of $i\varkappa_l$, the Green function $g(x,y,k)$ can be written as

$$g(x,y,k) = \frac{\sum_{r=1}^{m_l} \psi_r(x)\overline{\psi_r(y)}}{(k - i\varkappa_l)(k + i\varkappa_l)} + g_l(x,y,k)\,.$$

Here $g_l(x,y,k)$ is locally bounded and $\{\psi_r\}_{r=1}^{m_l}$ forms an orthonormal eigenbasis corresponding to the eigenvalue $\lambda_l = -\varkappa_l^2$. Hence,

$$\begin{aligned} \det X_{22}(y,k) &= (2ik)^{-n}e^{-inky} \det A^{-1}(k) \\ &= (-1)^n e^{-inkx} \det g(x,y,k) = O\left(|k - i\varkappa_l|^{-m_l}\right) \end{aligned}$$

as $k \to i\varkappa_l$. This implies that $\det A(k)$ has a zero of the order

$$m_l' \leq m_l$$

at $k = i\varkappa_l$. Finally, the last inequality and (1.5) imply (1.31).

1.6. **The matrix function $T(x,k)$.** Consider the matrix function

$$T(x,k) = \mathbf{1} + H(x,k) = \mathbf{1} + \int_x^\infty K(x,t,k)T(t,k)\,dt\,. \tag{1.36}$$

According to subsection 1.3 the matrix-valued function $T(x,k)$ is smooth and uniformly bounded for

$$(x,k) \in \mathbb{R} \times \{k \in \mathbb{C} : \mathrm{Im}\, k \geq 0 \quad \text{and} \quad |k| \geq 1\}\,.$$

Obviously $T(x,k) = 1$ for $x \geq x_{\max}$. Integrating by parts in (1.36) and using (1.18) we obtain

$$\frac{d^l}{dx^l}T(x,k) = -\int_x^\infty e^{2ik(t-x)}\frac{d^{l-1}}{dt^{l-1}}\left(V(t)T(t,k)\right)dt \tag{1.37}$$

for all $l \in \mathbb{N}$. Since supp $V \subseteq [x_{\min}, x_{\max}]$ we find

$$d^lT(x,k)/dx^l = 0 \qquad \text{as} \qquad x_{\max} \leq x\ , \tag{1.38}$$

$$\left\|d^lT(x,k)/dx^l\right\| \leq C_8 \qquad \text{as} \qquad x_{\min} \leq x \leq x_{\max}\ , \tag{1.39}$$

$$\left\|d^lT(x,k)/dx^l\right\| \leq C_9 e^{2(x-x_{\min})\mathrm{Im}\,k} \qquad \text{as} \qquad x \leq x_{\min}\ , \tag{1.40}$$

for all k with Im $k \geq 0$ and $|k| \geq 1$. The constants C_8 and C_9 depend only upon V, n and l. If we integrate the r.h.s. of (1.37) by parts, then (1.39) and (1.40) imply

$$\left\|d^lT(x,k)/dx^l\right\| \leq \frac{C_{10}}{1+|k|} \qquad \text{as} \qquad x_{\min} \leq x \leq x_{\max}\ , \tag{1.41}$$

$$\left\|d^lT(x,k)/dx^l\right\| \leq \frac{C_{11}}{1+|k|}e^{2(x-x_{\min})\mathrm{Im}\,k} \qquad \text{as} \qquad x \leq x_{\min}\ , \tag{1.42}$$

for all k with Im $k \geq 0$ and $|k| \geq 1$. The constants C_{10} and C_{11} depend only upon V, n and l.

In a similar way integrating by parts in (1.37), we obtain the asymptotical decompositions

$$\begin{aligned}
\frac{d^l}{dx^l}T(x,k) &= -\int_x^\infty e^{2ik(t-x)}\frac{d^{l-1}}{dt^{l-1}}(V(t)T(t,k))\,dt \\
&= \left\{\sum_{r=1}^{q}\frac{(-1)^{r+1}}{(2ik)^r}\frac{d^{r+l-2}}{dx^{r+l-2}}\right\}(V(x)T(x,k)) \\
&\quad + (-1)^{q+1}\int_x^\infty \frac{e^{2ik(t-x)}}{(2ik)^q}\frac{d^{q+l-1}}{dt^{q+l-1}}\left(V(t)T(t,k)\right)\,dt \\
&= \left\{\sum_{r=1}^{q-1}\frac{(-1)^{r+1}}{(2ik)^r}\frac{d^{r+l-2}}{dx^{r+l-2}}\right\}(V(x)T(x,k)) + R_{q,l}(x,k)
\end{aligned} \tag{1.43}$$

as $|k| \geq 1$, Im $k > 0$. Here

$$R_{q,l}(x,k) = 0 \qquad \text{as} \qquad x_{\max} \leq x\ , \tag{1.44}$$

$$\|R_{q,l}(x,k)\| \leq C_{12}(1+|k|)^{-q} \qquad \text{as} \qquad x_{\min} \leq x \leq x_{\max}\ , \tag{1.45}$$

$$\|R_{q,l}(x,k)\| \leq \frac{C_{13}}{(1+|k|)^q}e^{2(x-x_{\min})\mathrm{Im}\,k} \qquad \text{as} \qquad x \leq x_{\min}\ . \tag{1.46}$$

The constants C_{12} and C_{13} depend upon V, n, l and q.

Since $d^l H/dx^l = d^l T/dx^l$ for all $l \in \mathbb{N}$, integration by parts in (1.23) and the inequalities (1.38), (1.41) and (1.42) give (1.26).

1.7. **The matrix function** $\sigma(x,k)$**.** By using (1.16), (1.20) and Remark 1.1 for sufficiently large $|k|$, Im $k \geq 0$, the matrix $T(x,k) = \mathbb{1} + H(x,k)$ is invertible for all $x \in \mathbb{R}$ and

$$\|T^{-1}(x,k)\| \leq C_{14} \quad \text{for all} \quad x \in \mathbb{R}\,, \quad |k| > C_{15}\,, \quad \operatorname{Im} k \geq 0\,, \tag{1.47}$$

with sufficiently large constants $C_{14} = C_{14}(V,n)$ and $C_{15} = C_{15}(V,n)$. Hence, for sufficiently large $|k|$ with Im $k \geq 0$ the matrix function

$$\sigma(x,k) = \left[\frac{d}{dx} T(x,k)\right] T^{-1}(x,k) \tag{1.48}$$

is well defined for all $x \in \mathbb{R}$. Liouville's formula

$$\frac{d}{dx}(\ln \det T(x,k)) = \operatorname{tr}\left\{\left[\frac{d}{dx}T(x,k)\right] T^{-1}(x,k)\right\}$$

implies

$$\frac{d}{dx}\left(\ln \det e^{-ikx} F(x,k)\right) = \operatorname{tr} \sigma(x,k)\,.$$

Since $e^{-ikx}F(x,k) = \mathbb{1}$ as $x \geq x_{\max}$ and

$$e^{-ikx}F(x,k) = e^{-2ikx}B(k) + A(k) = A(k) + o(1)$$

as $x \to -\infty$, Im $k \geq \epsilon > 0$, we finally conclude that

$$\ln \det A(k) = -\int_{-\infty}^{+\infty} \operatorname{tr} \sigma(x,k)\, dx\,, \tag{1.49}$$

$$|k| \geq C_{15}\,, \qquad \operatorname{Im} k \geq \epsilon > 0\,.$$

Remark 1.2. Formula (1.49) is a matrix version of the corresponding well-known identity for scalar Schrödinger operators (see e.g. §3 in [8]).

1.8. **The asymptotical decomposition of** $\sigma(x,k)$**.** Next we shall develop $\sigma(x,k)$ into an asymptotical series with respect to the inverse powers of k. For the sake of future references we compute the first three terms, although we only need the second one in this paper.

If we apply (1.43) with $q = 2$, $l = 1$ we find that

$$\sigma = \frac{1}{2ik} V + Q_2\,, \qquad Q_2 = R_{2,1} T^{-1}\,, \tag{1.50}$$

while (1.43) with $q = 4$, $l = 1$ gives

$$\sigma = \frac{1}{(2ik)^3}\left\{\frac{d^2V}{dx^2} + 2\frac{dV}{dx}\sigma + V\frac{d^2T}{dx^2}T^{-1}\right\} - \frac{1}{(2ik)^2}\left\{\frac{dV}{dx} + V\sigma\right\} + \frac{1}{2ik}V + R_{4,1}T^{-1}\,. \tag{1.51}$$

Inserting (1.50) into (1.51) we obtain

$$\sigma = \frac{1}{2ik}V - \frac{1}{(2ik)^2}\frac{dV}{dx} - \frac{1}{(2ik)^3}\left\{V^2 - \frac{d^2V}{dx^2}\right\} + Q_4\,. \tag{1.52}$$

Finally, if we insert in a similar way (1.52) and (1.43) with $l = 2$, $q = 3$ as well as $l = 3$, $q = 2$ into (1.43) with $l = 1$ and $q = 6$, we arrive at

$$\begin{aligned}\sigma = {} & (2ik)^{-1}V - (2ik)^{-2}\frac{dV}{dx} + (2ik)^{-3}\left\{\frac{d^2V}{dx^2} - V^2\right\} \\ & - (2ik)^{-4}\left\{\frac{d^3V}{dx^3} - 2\frac{dV^2}{dx}\right\} \\ & + (2ik)^{-5}\left\{\frac{d^4V}{dx^4} - 3\frac{d^2V^2}{dx^2} + \left(\frac{dV}{dx}\right)^2 + 2V^3\right\} + Q_6\,.\end{aligned} \tag{1.53}$$

As well as $R_{q,l}$ the terms Q_2, Q_4 and Q_6 satisfy the inequalities of the type (1.44) – (1.46) with $q = 2$, $q = 4$, and $q = 6$, respectively. Then we conclude that

$$\int_{-\infty}^{+\infty} \operatorname{tr} Q_q(x,k)\,dx = O(|k|^{-q})\,, \qquad q = 2,4,6\,,$$

as $|k| \to \infty$ with $\operatorname{Im} k \ge \epsilon > 0$ and thus,

$$\begin{aligned}\int_{-\infty}^{+\infty} \operatorname{tr} \sigma(x,k)\,dx = {} & \frac{1}{2ik}\int_{-\infty}^{+\infty} \operatorname{tr} V\,dx - \frac{1}{(2ik)^3}\int_{-\infty}^{+\infty} \operatorname{tr} V^2\,dx \\ & + \frac{1}{(2ik)^5}\int_{-\infty}^{+\infty}\left[2\operatorname{tr} V^3 + \operatorname{tr}\left(\frac{dV}{dx}\right)^2\right] dx + O(|k|^{-6})\end{aligned} \tag{1.54}$$

as $|k| \to \infty$ with $\operatorname{Im} k \ge \epsilon > 0$.

1.9. **The dispersion formula.** Let

$$\{\lambda_l\}_{l=1}^N = \{(i\varkappa_l)^2\}_{l=1}^N, \qquad \varkappa_l > 0,$$

be the finite set of the negative eigenvalues of (1.1). Each eigenvalue occurs in this set only once. Let m_l be the order of zero of $\det A(k)$ at the point

$k = i\varkappa_l$, which by section 1.5 equals the multiplicity of the corresponding eigenvalue. Then the arguments in section 1.5 imply that the function

$$M(k) = \ln\left\{\det A(k)\prod_{l=1}^{N}\left(\frac{k+i\varkappa_l}{k-i\varkappa_l}\right)^{m_l}\right\} \tag{1.55}$$

is analytic for Im $k > 0$ and continuous up to the boundary except $k = 0$, where it has at most a logarithmic singularity. Moreover, the inequality (1.24) gives

$$|M(k)| \le C_2(V)|k|^{-1}$$

for all sufficiently large $|k| > C$, Im $k \ge 0$. Hence, by applying Cauchy's formula for large semi-circles in the upper half-plane we obtain

$$\int_{-\infty}^{+\infty}\frac{M(z)\,dz}{z-k} = (2\pi i)M(k)\,, \qquad \int_{-\infty}^{+\infty}\frac{M(z)\,dz}{z-\bar{k}} = 0$$

for arbitrary k with Im $k > 0$. This implies

$$M(k) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\text{Re}\ M(z)}{z-k}\,dz\,, \tag{1.56}$$

which by (1.55) is equivalent to

$$\ln\det A(k) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\ln|\det A(z)|\,dz}{z-k} + \sum_{l=1}^{N} m_l \ln\frac{k-i\varkappa_l}{k+i\varkappa_l} \tag{1.57}$$

for all k with Im $k > 0$.

1.10. **Trace formulae for elliptic systems.** Note that

$$\begin{aligned}\sum_{l=1}^{N} m_l \ln\frac{k-i\varkappa_l}{k+i\varkappa_l} &= \frac{2}{ik}\sum_{l=1}^{N} m_l\varkappa_l - \frac{2}{3ik^3}\sum_{l=1}^{N} m_l\varkappa_l^3 \\ &\quad + \frac{2}{5ik^5}\sum_{l=1}^{N} m_l\varkappa_l^5 + O(|k|^{-6})\end{aligned} \tag{1.58}$$

as $|k| \to \infty$, Im $k \ge \epsilon > 0$. On the other hand from (1.13) and (1.26) we have

$$\ln|\det A(z)| = 2^{-1}\ln|\det(\mathbf{1} + B(-z)B^*(-z))| = O(|z|^{-m}), \qquad z \in \mathbb{R}\,,$$

as $|z| \to \infty$, for all $m \in \mathbb{N}$. Hence, the integral in (1.57) permits the asymptotical decomposition

$$\int_{-\infty}^{+\infty} \frac{\ln|\det A(z)|\,dz}{z-k} = -\sum_{j=0}^{m} \frac{I_j}{k^{j+1}} + O(|k|^{m+1}),$$

$$I_j = \int_{-\infty}^{+\infty} z^j \ln|\det A(z)|\,dz \tag{1.59}$$

as $|k| \to \infty$, $\operatorname{Im} k \geq \epsilon > 0$.

Combining (1.58), (1.59) with $m = 5$ and (1.54) we obtain

$$\frac{1}{4}\int \operatorname{tr} V\,dx = \frac{I_0}{2\pi} - \sum_{l=1}^{N} m_l \varkappa_l, \tag{1.60}$$

$$\frac{3}{16}\int \operatorname{tr} V^2\,dx = \frac{3I_2}{2\pi} + \sum_{l=1}^{N} m_l \varkappa_l^3, \tag{1.61}$$

$$\frac{5}{32}\int \operatorname{tr} V^3\,dx + \frac{5}{64}\int \operatorname{tr}\left(\frac{dV}{dx}\right)^2 dx = \frac{5I_4}{2\pi} - \sum_{l=1}^{N} m_l \varkappa_l^5. \tag{1.62}$$

Finally we remark, that in view of (1.13)

$$I_j \geq 0 \tag{1.63}$$

for all even, non-negative integers j.

2. Sharp Lieb-Thirring inequalities for second order one-dimensional Schrödinger type systems

2.1. A Lieb-Thirring estimate for finite systems. Let us first consider the operator on the l.h.s. of (1.1) in $L^2(\mathbb{R},\mathbb{C}^n)$ for some smooth, compactly supported Hermitian matrix potential V. Preserving the notation of the previous section the bounds (1.61) and (1.63) imply

$$\operatorname{tr}\left(-\frac{d^2}{dx^2}\otimes \mathbf{1} + V(x)\right)_-^{3/2} = \sum_l m_l \varkappa_l^3 \leq \frac{3}{16}\int \operatorname{tr} V^2(x)\,dx. \tag{2.1}$$

By continuity (2.1) extends to all Hermitian matrix potentials, for which $\operatorname{tr} V^2$ is integrable. Finally, a standard variational argument allows one to replace V by its negative part V_-:

$$\operatorname{tr}\left(-\frac{d^2}{dx^2}\otimes \mathbf{1} + V(x)\right)_-^{3/2} \leq \frac{3}{16}\int \operatorname{tr} V_-^2(x)\,dx. \tag{2.2}$$

The constant in the r.h.s. of this inequality is sharp and coincides with the classical constant $L^{cl}_{3/2,1}$. In particular, this constant does not depend on the internal dimension n of the system.

2.2. **Operator-valued differential equations.** Let $\mathbf{G}$ be a separable Hilbert space with the scalar product $< \cdot, \cdot >_{\mathbf{G}}$ and the norm $\|\cdot\|_{\mathbf{G}}$. Let $H^1(\mathbb{R}, \mathbf{G})$ and $H^2(\mathbb{R}, \mathbf{G})$ be the Sobolev spaces of all functions

$$u(\cdot) : \mathbb{R} \to \mathbf{G}\,,$$

for which the respective norms

$$\|u\|_{H^1}^2 = \int_{-\infty}^{+\infty} \left(\|u'\|_{\mathbf{G}}^2 + \|u\|_{\mathbf{G}}^2\right)\, dx$$
$$\|u\|_{H^2}^2 = \int_{-\infty}^{+\infty} \left(\|u''\|_{\mathbf{G}}^2 + \|u\|_{\mathbf{G}}^2\right)\, dx$$

are finite. Finally, let $1_{\mathbf{G}}$ be the identity operator on $\mathbf{G}$. Then the operator $-\frac{d^2}{dx^2} \otimes 1_{\mathbf{G}}$ defined on $H^2(\mathbb{R}, \mathbf{G})$ is self-adjoint in $L^2(\mathbb{R}, \mathbf{G})$. It corresponds to the closed quadratic form

$$h[u, u] = \int \|u'\|_{\mathbf{G}}^2\, dx$$

with the form domain $H^1(\mathbb{R}, \mathbf{G})$.

Let $\mathcal{B}$ and $\mathcal{K}$ respectively be the spaces of all bounded and compact linear operators on $\mathbf{G}$. Let $\|\cdot\|_{\mathcal{B}}$ denote the corresponding operator norm. Consider an operator-valued function

$$W(\cdot) : \mathbb{R} \to \mathcal{B}\,,$$

for which $W(x) = (W(x))^*$, $x \in \mathbb{R}$ and $\|W(\cdot)\|_{\mathcal{B}} \in L^p(\mathbb{R})$, $1 < p < \infty$. Denote

$$w[u, u] = \int_{-\infty}^{+\infty} \langle W(x)u(x), u(x)\rangle_{\mathbf{G}}\, dx\,.$$

This form is well-defined on $H^1(\mathbb{R}, \mathbf{G})$ and

$$|w[u, u]| \le C_{16} \left(\int_{-\infty}^{+\infty} \|W(x)\|_{\mathcal{B}}^p dx\right)^{1/p} \|u\|_{H^1}^2\,. \tag{2.3}$$

The constant C_{16} does not depend upon W or u. Moreover, for all $\epsilon > 0$ there exists a finite constant $C_{17}(\epsilon, W)$, such that

$$|w[u, u]| \le \epsilon h[u, u] + C_{17}(\epsilon, W) \int \|u\|_{\mathbf{G}}^2\, dx\,. \tag{2.4}$$

Both (2.3) and (2.4) follow immediately from the corresponding inequalities which hold in the scalar case. Hence, the quadratic form

$$h[u, u] + w[u, u]$$

is semi-bounded from below and closed on $H^1(\mathbb{R}, \mathbf{G})$. It induces a self-adjoint semi-bounded operator

$$Q = -\frac{d^2}{dx^2} \otimes \mathbf{1}_{\mathbf{G}} + W(x) \tag{2.5}$$

on $L^2(\mathbb{R}, \mathbf{G})$.

If in addition $W(x) \in \mathcal{K}$ for a.e. $x \in \mathbb{R}$, then the form $w[\cdot,\cdot]$ is relative compact with respect to the metric on $H^1(\mathbb{R}, \mathbf{G})$. In order to prove this fact we introduce the orthogonal projections P_M on the linear span of the first M elements of some fixed orthonormal basis in $\mathbf{G}$. As a consequence, the Birman-Schwinger principle implies, that the negative spectrum of the operator Q is discrete and might accumulate only to zero. In other words, the operator Q_- is compact on $L^2(\mathbb{R}, \mathbf{G})$.

2.3. **A Lieb-Thirring estimate for operator-valued differential equations.** We shall prove the following Theorem:

Theorem 2.1. *Let $W(x)$ be self-adjoint Hilbert-Schmidt operators on $\mathbf{G}$ for a.e. $x \in \mathbb{R}$ and let $\operatorname{tr} W^2(\cdot) \in L^1(\mathbb{R}, \mathbf{G})$. Then we have*

$$\operatorname{tr} \left(-\frac{d^2}{dx^2} \otimes \mathbf{1}_{\mathbf{G}} + W(x)\right)_-^{3/2} \leq L^{\mathrm{cl}}_{3/2,1} \int_{-\infty}^{+\infty} \operatorname{tr} W_-^2 \, dx \,, \tag{2.6}$$

where according to (0.4) *it holds $L^{\mathrm{cl}}_{3/2,1} = 3/16$.*

Proof. Assume that (2.6) fails. Then there exists a non-positive operator family W satisfying $\operatorname{tr} W^2(\cdot) \in L^1(\mathbb{R})$ and some sufficiently small $\epsilon > 0$, such that

$$\operatorname{tr} \chi_\epsilon^{3/2}(Q) > \frac{3}{16} \int_{-\infty}^{+\infty} \operatorname{tr} W^2 \, dx \,. \tag{2.7}$$

Here

$$\chi_\epsilon(Q) = -E_{(-\infty,-\epsilon)}(Q) Q \,,$$

with $E_{(-\infty,-\epsilon)}(Q)$ being the spectral projection of Q onto the interval $(-\infty, -\epsilon)$. Since Q_- is compact, the operator $E_{(-\infty,-\epsilon)}(Q)$ is of a finite rank $n(\epsilon)$.

Fix some orthonormal basis in $\mathbf{G}$ and let $\mathbb{P}_M$ be the projection on the linear span of its first M elements. Consider the auxiliary operators

$$Q(M, \epsilon) = E_{(-\infty,-\epsilon)}(Q)(1(x) \otimes P_M) Q (1(x) \otimes P_M) E_{(-\infty,-\epsilon)}(Q) \,.$$

Obviously we have $\operatorname{rank} Q(M, \epsilon) \leq n(\epsilon)$ for all M. Since $1(x) \otimes \mathbb{P}_M$ turns to the identity operator on $L^2(\mathbb{R}, \mathbf{G})$ in the strong operator topology

as $M \to \infty$, then the operators $Q(M,\epsilon)$ converge to $\chi_\epsilon(Q)$ in the $L^2(\mathbb{R}\mathbf{G})$ operator norm, as $M \to \infty$ and

$$\operatorname{tr}\,(Q(M,\epsilon))_-^{3/2} \to \operatorname{tr}\chi_\epsilon(Q) \quad \text{as} \quad M \to \infty\,.$$

Thus,

$$\operatorname{tr}\,(Q(M,\epsilon))_-^{3/2} > \frac{3}{16}\int_{-\infty}^{+\infty} \operatorname{tr} W^2\,dx \tag{2.8}$$

for some sufficiently large M. On the other hand, a standard variational argument implies

$$\operatorname{tr}\,(Q(M,\epsilon))_-^{3/2} \le \operatorname{tr}\,\left((1(x)\otimes P_M)Q(1(x)\otimes P_M)\right)_-^{3/2}\,.$$

Observe that the expression on the r.h.s. is nothing else but the Riesz mean of the order $\gamma = 3/2$ of the negative eigenvalues of the $M\times M$-system (1.1) with $V(x) = P_M W(x) P_M$. Thus, from (2.2) we obtain

$$\operatorname{tr}\,(Q(M,\epsilon))_-^{3/2} \le \frac{3}{16}\int \operatorname{tr} V^2(x)\,dx \le \frac{3}{16}\int \operatorname{tr} W^2(x)\,dx\,,$$

which contradicts (2.8). This completes the proof.

2.4. **Lieb-Thirring estimates for Riesz means of negative eigenvalues of the order $\gamma \ge 3/2$.** We shall now suppose, that the non-positive operator family $W(x)$ satisfies

$$\operatorname{tr} W_-^{\gamma+\frac12}(x) \in L^1(\mathbb{R}) \quad \text{for some} \quad \gamma > 3/2\,. \tag{2.9}$$

Let $dE_{(-\infty,\lambda)}(Q)$ be the spectral measure of the operator Q. Repeating the arguments of Aizenman and Lieb [1], we find

$$\begin{aligned}
B\left(\gamma-\frac32,\frac52\right)\operatorname{tr} Q_-^{\gamma} &= \operatorname{tr}\,\left\{\int_{-\infty}^{0} dE_{(-\infty,\lambda)}(Q)\int_0^\infty t^{\gamma-\frac52}(t+\lambda)_-^{3/2}\,dt\right\}\\
&= \int_0^\infty t^{\gamma-\frac52}\operatorname{tr}\,(Q+t)_-^{3/2}\,dt\\
&\le \frac{3}{16}\int_0^\infty dt\, t^{\gamma-\frac52}\int_{-\infty}^{+\infty}\operatorname{tr}\,(W(x)+t)_-^2\,dx\,,
\end{aligned}$$

where $B(x,y) = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$ is the Beta function. Let $-\mu_j(x) < 0$ be the negative eigenvalues of $W(x)$. Then

$$\begin{aligned}
\int_0^\infty dt\, t^{\gamma-\frac{5}{2}} \int_{-\infty}^{+\infty} \operatorname{tr}\,(W(x)+t)_-^2\, dx \\
&= \sum_{j=1}^\infty \int_{-\infty}^{+\infty} dx \int_0^\infty dt\, t^{\gamma-\frac{5}{2}} (t-\mu_j(x))_-^2 \\
&= B\left(\gamma-\frac{3}{2}, 3\right) \int_{-\infty}^{+\infty} dx \sum_{j=1}^\infty \mu_j^{\gamma+\frac{1}{2}}(x) \\
&= B\left(\gamma-\frac{3}{2}, 3\right) \int_{-\infty}^{+\infty} \operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x)\, dx\,.
\end{aligned}$$

From (0.4) we obtain

$$L_{\gamma,1}^{\mathrm{cl}} = \frac{\Gamma(\gamma+1)}{2\pi^{1/2}\Gamma(\gamma+\frac{3}{2})} = \frac{3}{16}\cdot\frac{\Gamma(\gamma+1)\Gamma(3)}{\Gamma(\gamma+\frac{3}{2})\Gamma(\frac{5}{2})} = \frac{3}{16}\cdot\frac{B\left(\gamma-\frac{3}{2},3\right)}{B\left(\gamma-\frac{3}{2},\frac{5}{2}\right)}\,,$$

and this implies

Theorem 2.2. *Let the non-positive operator family* $W(x)$ *satisfy* (2.9). *Then*

$$\operatorname{tr}\left(-\frac{d^2}{dx^2}\otimes \mathbf{1}_{\mathbf{G}} + W(x)\right)_-^\gamma \le L_{\gamma,1}^{\mathrm{cl}} \int_{-\infty}^{+\infty} \operatorname{tr} W_-^{\gamma+\frac{1}{2}}(x)\, dx\,. \tag{2.10}$$

It remains to note, that the constant $L_{\gamma,1}^{\mathrm{cl}}$ in (2.10) is approached for potentials αW as $\alpha \to +\infty$.

3. Lieb-Thirring estimates with sharp constants for Schrödinger operators in higher dimensions

3.1. **Lieb-Thirring estimates for Schrödinger operators.** Let $\mathbf{G}$ be a separable Hilbert space. We consider the operator

$$-\Delta\otimes\mathbf{1}_{\mathbf{G}} + V(x) \tag{3.1}$$

in $L^2(\mathbb{R}^d, \mathbf{G})$. If the family

$$V(\cdot) : \mathbb{R}^d \to \mathcal{B}$$

of bounded self-adjoint operators on $\mathbf{G}$ satisfies

$$\|V(\cdot)\|_{\mathcal{B}} \in L^p(\mathbb{R}^d), \qquad \max\{1, d/2\} < p \le \infty\,, \tag{3.2}$$

then the quadratic form

$$v[u,u] = \int_{\mathbb{R}^d} \langle V(x)u(x), u(x)\rangle_{\mathbf{G}}\, dx$$

is zero-bounded with respect to

$$h[u,u] = \int_{\mathbb{R}^d} \sum_{j=1}^{d} \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbf{G}}^2 dx\,.$$

This immediately follows from the corresponding scalar result and the arguments given when proving the inequalities (2.3), (2.4). Hence the quadratic form $h[\cdot,\cdot] + v[\cdot,\cdot]$ is semi-bounded from below, closed on the Sobolev space $H^1(\mathbb{R}^d, \mathbf{G})$ and thus generates the operator (3.1). As in subsection 3.2 one can show, that if in addition to (3.2) we have $V(x) \in \mathcal{K}$ for a.e. $x \in \mathbb{R}^d$, then the negative spectrum of the operator (3.1) is discrete.

The main result of this paper is

Theorem 3.1. *Assume that* $V(x) \leq 0$ *for a.e.* $x \in \mathbb{R}^d$ *and that* tr $V^{\frac{d}{2}+\gamma}(\cdot)$ *is integrable for some* $\gamma \geq 3/2$. *Then*

$$\text{tr } (-\Delta \otimes \mathbf{1}_{\mathbf{G}} + V(x))_-^{\gamma} \leq L^{\mathrm{cl}}_{\gamma,d} \int_{\mathbb{R}^d} \text{tr } V_-^{\frac{d}{2}+\gamma}(x)\, dx\,. \tag{3.3}$$

Proof. We use the induction arguments with respect to d. For $d = 1$, $\gamma \geq 3/2$ the bound (3.3) is identical to (2.10). Assume that we have (3.3) for $d-1$ and all $\gamma \geq 3/2$. Consider the operator (3.1) in the (external) dimension d. We rewrite the quadratic form $h[u,u] + v[u,u]$ for $u \in H^1(\mathbb{R}^d, \mathbf{G})$ as follows

$$\begin{aligned}
h[u,u] + v[u,u] &= \int_{-\infty}^{+\infty} h(x_d)[u,u]\, dx_d + \int_{-\infty}^{+\infty} w(x_d)[u,u]\, dx_d\,, \\
h(x_d)[u,u] &= \int_{\mathbb{R}^{d-1}} \left\| \frac{\partial u}{\partial x_d} \right\|_{\mathbf{G}}^2 dx_1 \cdots x_{d-1}\,, \\
w(x_d)[u,u] &= \int_{\mathbb{R}^{d-1}} \left[\sum_{j=1}^{d-1} \left\| \frac{\partial u}{\partial x_j} \right\|_{\mathbf{G}}^2 + \langle V(x)u, u\rangle_{\mathbf{G}} \right] dx_1 \cdots x_{d-1}\,.
\end{aligned}$$

The form $w(x_d)$ is closed on $H^1(\mathbb{R}^{d-1}, \mathbf{G})$ for a.e. $x_d \in \mathbb{R}$ and it induces the self-adjoint operator

$$W(x_d) = -\sum_{k=1}^{d-1} \frac{\partial^2}{\partial x_k^2} \otimes \mathbf{1}_{\mathbf{G}} + V(x_1, \ldots, x_{d-1}; x_d)$$

on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$. The negative spectrum of this $(d-1)$–dimensional Schrödinger system is discrete, hence $W_-(x_d)$ is compact on $L^2(\mathbb{R}^{d-1}, \mathbf{G})$

and according to our induction hypothesis $\operatorname{tr} W_-^{\gamma+\frac12}(x_d)$ satisfies the inequality

$$\operatorname{tr} W_-^{\gamma+\frac12}(x_d) \le L^{cl}_{\gamma+\frac12,d-1}\int_{\mathbb{R}^{d-1}} \operatorname{tr} V_-^{\gamma+\frac{d}{2}}(x_1,\dots,x_{d-1};x_d)\,dx_1\cdots dx_{d-1} \tag{3.4}$$

for a.e. $x_d \in \mathbb{R}$. For $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^{d-1})$, the function $\operatorname{tr} W_-^{\gamma+\frac12}(\cdot)$ is integrable.

Let $w_-(x_d)[\cdot,\cdot]$ be the quadratic form corresponding to the operator $W_-(x_d)$ on $\mathbf{H} = L^2(\mathbb{R}^{d-1},\mathbf{G})$. Then we have $w(x_d)[u,u] \ge -w_-(x_d)[u,u]$ and

$$h[u,u]+v[u,u] \ge \int_{-\infty}^{+\infty}\left[\left\|\frac{\partial u}{\partial x_d}\right\|^2_{\mathbf{H}} - \langle W_-(x_d)u,u\rangle_{\mathbf{H}}\right] dx_d \tag{3.5}$$

for all $u \in H^1(\mathbb{R}^d,\mathbf{G})$. According to section 2.2 the form on the r.h.s. of (3.5) can be closed to $H^1(\mathbb{R},\mathbf{H})$ and induces the self-adjoint operator

$$-\frac{d^2}{dx_d^2}\otimes \mathbf{1}_{\mathbf{H}} - W_-(x_d)$$

on $L^2(\mathbb{R},\mathbf{H})$. Then (3.5) implies

$$\operatorname{tr}(-\Delta\otimes\mathbf{1}_{\mathbf{G}}+V)_-^\gamma \le \operatorname{tr}\left(-\frac{d^2}{dx_d^2}\otimes\mathbf{1}_{\mathbf{H}} - W_-(x_d)\right)_-^\gamma . \tag{3.6}$$

We can now apply (2.10) to the r.h.s. of (3.6) and in view of (3.4) we find

$$\begin{aligned}\operatorname{tr}\left(-\frac{d^2}{dx_d^2}\otimes\mathbf{1}_{\mathbf{H}} - W_-(x_d)\right)_-^\gamma &\le L^{cl}_{\gamma,1}\int_{-\infty}^{+\infty}\operatorname{tr} W_-^{\gamma+\frac12}(x_d)\,dx_d\\ &\le L^{cl}_{\gamma,1}L^{cl}_{\gamma+\frac12,d-1}\int_{\mathbb{R}^d}\operatorname{tr} V_-^{\gamma+\frac{d}{2}}(x)\,dx\,.\end{aligned}$$

The calculation

$$\begin{aligned}L^{cl}_{\gamma,1}L^{cl}_{\gamma+\frac12,d-1} &= \frac{\Gamma(\gamma+1)}{2\pi^{\frac12}\Gamma(\gamma+\frac12+1)}\cdot\frac{\Gamma(\gamma+\frac12+1)}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma(\gamma+\frac12+\frac{d-1}{2}+1)}\\ &= \frac{\Gamma(\gamma+1)}{2^d\pi^{\frac{d}{2}}\Gamma(\gamma+\frac{d}{2}+1)} = L^{cl}_{\gamma,d}\end{aligned}$$

completes the proof.

For the special case $\mathbf{G}=\mathbb{C}$ we obtain the Lieb-Thirring bounds for scalar Schrödinger operators with the (sharp) classical constant $L_{\gamma,d} = L^{cl}_{\gamma,d}$ for $\gamma \ge 3/2$ in all dimensions d.

3.2. **Lieb-Thirring estimates for magnetic operators.** Following a remark by B. Helffer [10] we demonstrate, how Theorem 3.1 can be extended to Schrödinger operators with magnetic fields. Let

$$\mathbf{a}(x) = (a_1(x), \dots, a_d(x))^t, \qquad d \geq 2\,,$$

be a magnetic vector potential with real-valued entries $a_k \in L^2_{loc}(\mathbb{R}^d)$. Put

$$H(\mathbf{a}) = (i\nabla + \mathbf{a}(x))^2 \otimes \mathbf{1}_{\mathbf{G}}\,.$$

Its form domain $d[h(\mathbf{a})]$ consists of the closure of all smooth compactly supported functions with respect to $h(\mathbf{a})[\cdot,\cdot] + \|\cdot\|^2_{L^2(\mathbb{R}^d,\mathbf{G})}$ (cf. [25]), where

$$h(\mathbf{a})[u,u] = \sum_{k=1}^{d} \int_{\mathbb{R}^d} \left\| \left(i\frac{\partial}{\partial x_k} + a_k \right) u \right\|^2_{\mathbf{G}} dx\,.$$

Let the operator family V and the corresponding form ν be defined as in the previous subsection. If (3.2) is satisfied, then one can apply Kato's inequality [14, 25], and find that the form

$$q(\mathbf{a})[u,u] = h(\mathbf{a})[u,u] + \nu[u,u] \tag{3.7}$$

is closed on $d[q(\mathbf{a})] = d[h(\mathbf{a})]$ and induces the self-adjoint operator

$$Q(\mathbf{a}) = H(\mathbf{a}) + V(x) \tag{3.8}$$

on $L^2(\mathbb{R}^d, \mathbf{G})$. Finally, by applying Kato's inequality to the higher-dimensional analog of (2.3) we see, that $V(x) \in \mathcal{K}$ for a.e. $x \in \mathbb{R}^d$ in conjunction with (3.2) implies that $Q(\mathbf{a})$ has discrete negative spectrum.

Theorem 3.2. *Assume that* $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$ *is a real vector field, and that the non-positive operator family* $V(x)$ *satisfies* $\operatorname{tr} V^{\frac{d}{2}+\gamma} \in L^1(\mathbb{R}^d)$ *for some* $\gamma \geq 3/2$*. Then*

$$\operatorname{tr}\,(H(\mathbf{a}) + V(x))^{\gamma}_{-} \leq L^{\mathrm{cl}}_{\gamma,d} \int_{\mathbb{R}^d} \operatorname{tr} V_{-}^{\frac{d}{2}+\gamma}\, dx\,. \tag{3.9}$$

Proof. In the dimension $d = 1$, any magnetic field can be removed by gauge transformation. Thus (2.10) can serve to initiate the induction procedure.

Assume now that (3.9) is known for all $\gamma \geq 3/2$ for the dimension $d - 1$ and consider the operator $H(\mathbf{a})$ in the dimension d. Put

$$W(x_d) = \left[\sum_{n=1}^{d-1} \left(i\frac{\partial}{\partial x_n} + a_n(x) \right)^2 \right] + V(x)\,.$$

We find that

$$q(\mathbf{a})[u,u] = \int_{\mathbb{R}^d} \left\| \left(i\frac{\partial}{\partial x_d} + a_d \right) u \right\|_{\mathbf{G}}^2 dx + \int_{\mathbb{R}} \langle W(x_d)u, u\rangle_{\mathbf{H}}\, dx_d$$
$$\geq \int_{\mathbb{R}^d} \left\| \left(i\frac{\partial}{\partial x_d} + a_d \right) u \right\|_{\mathbf{G}}^2 dx - \int_{\mathbb{R}} \langle W_-(x_d)u, u\rangle_{\mathbf{H}}\, dx_d\,,$$

where for fixed $x_d \in \mathbb{R}$ the operator $W_-(x_d)$ is the negative part of $W(x_d)$ on $\mathbf{H} = L^2(\mathbb{R}^{d-1}, \mathbf{G})$. We now choose a gauge in which a_d vanishes. Namely, put

$$\phi(x_1, \dots, x_d) = \int_0^{x_d} a_d(x_1, \dots, x_{d-1}, \tau)\, d\tau$$

and $\tilde{u}(x) = e^{i\phi(x)}u(x)$ for all $u \in d[q(\mathbf{a})]$. Then

(3.10)

$$q(\mathbf{a})[u,u] \geq \int_{\mathbb{R}^d} \left\| \frac{\partial \tilde{u}}{\partial x_d} \right\|_{\mathbf{G}}^2 dx - \int_{\mathbb{R}} \left\langle \tilde{W}(x_d)\tilde{u}, \tilde{u} \right\rangle_{\mathbf{H}} dx_d\,, \qquad u \in d[q(\mathbf{a})]\,,$$

where

$$\widetilde{W}(x_d) = e^{i\phi(x', x_d)} W_-(x_d) e^{-i\phi(x', x_d)}\,, \qquad x' = (x_1, \dots, x_{d-1})\,,$$

acts on $\mathbf{H}$ for any fixed $x_d \in \mathbb{R}$. Closing the form on the r.h.s. of (3.10) we see that

$$\text{tr}\ (H(\mathbf{a}) + V(x))_-^\gamma \leq \text{tr} \left(-\frac{d^2}{dx_d^2} \otimes \mathbf{1}_{\mathbf{H}} - \widetilde{W}(x_d) \right)_-^\gamma, \tag{3.11}$$

where the operator on the r.h.s. acts in $L^2(\mathbb{R}, \mathbf{H})$. From our induction hypothesis we have

$$\text{tr}\, \widetilde{W}^{\gamma+\frac{1}{2}}(x_d) = \text{tr}\, W_-^{\gamma+\frac{1}{2}}(x_d) \leq L^{\text{cl}}_{\gamma+\frac{1}{2}, d-1} \int_{\mathbb{R}^{d-1}} \text{tr}\, V_-^{\gamma+\frac{d}{2}}(x'; x_d)\, dx'\,.$$

Hence (2.10) can be applied to estimate the r.h.s. of (3.11) and we complete the proof of (3.9) in the same manner as in the proof of Theorem 3.1.

3.3. **Lieb-Thirring estimates for the Pauli operator.** As an application of Theorem 3.2 we deduce a Lieb-Thirring type bound for the Pauli operator. Preserving the notations of the previous subsection we put $d = 3$ and $\mathbf{G} = \mathbb{C}^2$. Let $\mathbf{a}(x) = (a_1(x), a_2(x), a_3(x))^t$ be a twice continuously differentiable vector function with real-valued entries. The Pauli operator is given by the differential expression

$$Z = Q(\mathbf{a}) \otimes \mathbf{1} + \begin{pmatrix} a_{1,2} & -ia_{3,1} + a_{2,3} \\ ia_{3,1} + a_{2,3} & -a_{1,3} \end{pmatrix} + V \otimes \mathbf{1}, \tag{3.12}$$

where $\mathbb{1}$ is the identity on $\mathbb{C}^2$, $V = V(x)$ is the multiplication by a real-valued scalar potential and

$$a_{j,k} = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}, \qquad k, j = 1, 2, 3\,.$$

Let $B(x)$ be the length of the vector $\mathcal{B}(x) = \operatorname{curl} \mathbf{a}(x)$. Then the two eigenvalues of the perturbation of the term $Q(\mathbf{a}) \otimes \mathbb{1}$ in (3.12) at some point $x \in \mathbb{R}^3$ are given by

$$V(x) \pm B(x).$$

If $V, B \in L^{\gamma+\frac{3}{2}}(\mathbb{R}^3)$ for some $\gamma \geq 3/2$, then Theorem 3.2 implies

$$\operatorname{tr} Z_-^\gamma \leq L^{\mathrm{cl}}_{\gamma,3} \left(\int \left\{ (V+B)_-^{\gamma+\frac{3}{2}} + (V-B)_-^{\gamma+\frac{3}{2}} \right\} dx \right). \tag{3.13}$$

3.4. **Acknowledgements.** The authors wish to express their gratitude to V.S. Buslaev and L.D. Faddeev for the useful discussions concerning trace formulae and B. Helffer for his valuable comments on magnetic Schrödinger operators. The first author has been supported by the Swedish Natural Sciences Research Council, Grant M-AA/MA 09364-320. The second author has been supported by the Swedish Natural Science Council dnr 11017-303. Partial financial support from the European Union through the TMR network FMRX-CT 96-0001 is gratefully acknowledged.

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